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ONE POSSIBLE SIMPLIFICATION OF THE DYNAMICAL EQUATION GOVERNING THE EVOLUTION OF ELLIPTICAL ACCRETION DISCS

Dimitar Dimitrov

Space Research Institute - Bulgarian Academy of Sciences

Abstract

It is shown that, under the assumption of the viscosity law $\eta = \beta\Sigma^{n}$, the integrals involved in the equation describing the dynamics of the disc, may be replaced by polynomials and exponential functions of the eccentricity e, its derivative $\dot{e} = \partial e/\partial(\ln p)$ with respect to the focal parameter p, and the power index n. This transformation is useful for numerical solving of the dynamical second-order differential equation, because it avoids numerical evaluation of the integrals and, possibly, contributes to a more stable computational procedure. Our consideration of the problem is limited to the case when the values of the parameter n are not integers.

Observations and theoretical studies give evidences that the accretion discs around compact objects (in the Newtonian approach) are not only circular in shape, but may also have elongated structure. In the later case it is possible the eccentricity e of the particle orbits to vary with the focal parameter p $(e = e(p))$; e.g., the outer parts of the disc are more elongated than the inner ones. We shall consider smooth accretion discs in the sense that the possible spiral structures into the disc are not taken into account. The present paper is based on the theory of elliptical accretion discs developed by Lyubarskij et al. [1] and in what follows we shall use their approach and notations. These authors have obtained the dynamical equation governing the motion of particles alone elliptical streamlines and determining the functional dependence $e = e(p)$ for *a priori* assumed viscosity law $\eta = \beta\Sigma^{n}$. Here η is the viscosity coefficient, $\Sigma = \Sigma(p)$ is the disc surface density, β and n are parameters independent of p. They have also solved this equation (using numerical methods) for some values of the power index n.

The case of constant eccentricity e (when e does not depend on p and azimuthal angle φ for all points of the accretion disc) is a particular case of the set of solutions of the dynamical equation. It was treated in details in [2]. In this paper we concentrate on the case $e = e(p)$ and show that the dynamical equation may be simplified to some extend, avoiding numerical computation of the integrals involved in it. Following Lyubarskij et al. [1], we introduce a new variable $u = \ln p$ and write $e = e(u)$ instead of $e = e(p)$. Correspondingly, we denote by $\dot{e}$ the derivative $\dot{e} = \partial e/\partial u$. The streamlines of the fluid particles are described by the equation [1]:

(1) $$[\,\mathrm{Y}\,(\partial \mathrm{Z}/\partial \dot{e}) - \mathrm{Z}\,(\partial \mathrm{Y}/\partial \dot{e})\,]\,\ddot{e} + [\,\mathrm{Y}\,(\partial \mathrm{Z}/\partial e) - \mathrm{Z}\,(\partial \mathrm{Y}/\partial e) - \mathrm{Y}^2 e\,]\,\dot{e} + \mathrm{Y}\,[\,(3/2)\mathrm{W} - \mathrm{Z} - (1/2)(1 - e^2)\,\mathrm{Y}\,] = 0\,.$$

In the above equation the auxiliary functions Y, Z and W (angle averaging with respect to φ has already been performed) are represented by the relations:

(2) $$3\mathrm{Y}(e, \dot{e}, n) = (1/2\pi)(p/GM)^{n/2}\,[\,(3 + e^2 + 2e\dot{e})\mathrm{I}_0 + (7e + e^3 - 4\dot{e} - 2e^3\dot{e})\mathrm{I}_1 + (4e^2 - 8e\dot{e})\mathrm{I}_2\,]\,,$$

(3) $$3\mathrm{Z}(e, \dot{e}, n) = (1/2\pi)(p/GM)^{n/2}\,[\,(3 + e^4 - 2e\dot{e} - 2e^3\dot{e})\mathrm{I}_0 + (13e + 2e^3 + e^5 - 4\dot{e} - 6e^2\dot{e} - 2e^4\dot{e})\mathrm{I}_1 + (22e^2 + 2e^4 - 12e\dot{e} - 4e^3\dot{e})\mathrm{I}_2 + (16e^3 - 12e^2\dot{e})\mathrm{I}_3 + (4e^4 - 4e^3\dot{e})\mathrm{I}_4\,]\,,$$

(4) $$9\mathrm{W}(e, \dot{e}, n) = (1/2\pi)(p/GM)^{n/2}\,[\,(9 - 2e^2 + e^4 + 4e\dot{e} - 4e^3\dot{e} + 8\dot{e}^2 + 4e^2\dot{e}^2)\mathrm{I}_0 + (33e - 2e^3 + e^5 - 24\dot{e} + 4e^2\dot{e} - 4e^4\dot{e} + 8e\dot{e}^2 + 4e^3\dot{e}^2)\mathrm{I}_1 + (48e^2 - 72e\dot{e} + 8\dot{e}^2)\mathrm{I}_2 + (32e^3 - 72e^2\dot{e} + 24e\dot{e}^2)\mathrm{I}_3 + (8e^4 - 24e^3\dot{e} + 16e^2\dot{e}^2)\mathrm{I}_4\,]\,.$$

In such a way, at the very beginning of the problem for finding the dependence $e = e(p)$ by solving the dynamical equation (1), there arises a difficulty due to the inevitable appearance of 7 integrals defined as:

(5) $$\mathrm{I}_k(e, \dot{e}, n) = \int_0^{2\pi} \cos^k \varphi\,(1 + e\cos\varphi)^{n-2}\,[1 + (e - \dot{e})\cos\varphi]^{-n-1}\,d\varphi\,,$$

$$k = 0, 1, \ldots, 4\,,$$

$$(6) \qquad I_{0-}(e, \dot{e}, n) = \int_0^{2\pi} (1 + e\cos\varphi)^{n-3}\,[1 + (e - \dot{e})\cos\varphi]^{-n-1}\, d\varphi\,,$$

$$(7) \qquad I_{0+}(e, \dot{e}, n) = \int_0^{2\pi} (1 + e\cos\varphi)^{n-2}\,[1 + (e - \dot{e})\cos\varphi]^{-n-2}\, d\varphi\,.$$

Integration over φ describes the angle averaging along the streamlines. Coefficients of the equation (1) depend on $e = e(u \equiv \ln p)$, $\dot{e} = \dot{e}(u \equiv \ln p)$ and the power index n, which is assumed to be independent of p and φ. The above integrals (5) - (7) are considered for values of $e(u)$ and $\dot{e}(u)$ which satisfy the restriction $|e - \dot{e}| < 1$, so no singularities arise during the integration. This requirement is connected to the condition that the metric in the curvilinear coordinates (p, φ) must be nonsingular and self-adjoint orbits do not intersect. Finding the solution of equation (1) is complicated by the fact that the unknown function $e(u)$ and its derivative $\dot{e}(u)$ enter into the integrands of (5) - (7). Substituting expressions (2) - (7) into the dynamical equation (1), describing the structure of the stationary accretion disc, leads to the following general form of this equation:

(8) $\quad \sum[A_{ik}(e, \dot{e}, n)\,\ddot{e} + B_{ik}(e, \dot{e}, n)]\, I_i(e, \dot{e}, n) I_k(e, \dot{e}, n) = 0$, where the sum is over i and k (i, k = 0-, 0+, 0, 1, . . . , 4), and i is less or equal to k.

Functions A_{ik} and B_{ik} are polynomials in e, $\dot{e}$ and n. In the present paper we show that the integrals (5) - (7) can also be expressed as polynomials and exponential functions of e, $\dot{e}$ and n, avoiding in such a way the numerical integrations during the procedure of numerical solution of (8).

Using the identities $1 = \cos^2\varphi + \sin^2\varphi$, $1 = (1 + e\cos\varphi) - e\cos\varphi$, $1 = [1 + (e - \dot{e})\cos\varphi\,] - (e - \dot{e})\cos\varphi$ and integrating by parts, we can find relations which enable us to eliminate (in principle) the integrals I_4, I_2, I_1 and I_0. For example, for the first two integrals we have:

$$(9) \qquad (e - \dot{e})eI_4 = (n - 2)\dot{e}(e^2 - 1)e^{-3}\, I_{0-} - (n - 2)\dot{e}(e^2 - 1)e^{-3}\, I_0 + [2e + (n - 2)\dot{e}(e^2 - 1)e^{-2}]\, I_1 + (n - 2)\dot{e}e^{-1} I_2 - [3e + (n - 2)\dot{e}]\, I_3\,,$$

(10) $\quad (e - \dot{e})eI_2 = [3e + (n - 2)\dot{e}\,]I_1 + [2e^4 - 3e^2 + (-2n + 4)e\dot{e} - 4e^3\dot{e} + (n - 2)\dot{e}^2 + 2e^2\dot{e}^2] \times [(e - \dot{e})e]^{-1} I_0 + (n - 2)(e - \dot{e})(e^2 - 1)e^{-1} I_{0-} + (n + 1)e(1 - e^2 + 2e\dot{e} - \dot{e}^2)(e - \dot{e})^{-1}\, I_{0+}$.

Further this approach *is not* appropriate to obtain solutions for $I_{0-}(e, \dot{e}, n)$ and $I_{0+}(e, \dot{e}, n)$, because another integrals (different from the system of the 7 integrals (5) - (7) must be involved; i.e., eliminating I_{0-} and I_{0+}, we would introduce new unknown integrals. It is reasonable then to use the derivatives of these integrals with respect to e and $\dot{e}$. Here we consider e, $\dot{e}$ and n as independent variables, because the analytical solutions $e = e(u)$ and $\dot{e} = \dot{e}(u)$ are so far unknown for us. We shall write the expressions for some of these derivatives:

$$(11) \quad \partial I_0/\partial e = (n-2)e^{-1}I_0 - (n-2)e^{-1}I_{0-} + (n+1)(e-\dot{e})^{-1}I_{0+} - (n+1)(e-\dot{e})^{-1}{}_0$$

$$(12) \quad \partial I_0/\partial \dot{e} = (n+1)(e-\dot{e})^{-1}I_0 - (n+1)(e-\dot{e})^{-1}I_{0+} ,$$

$$(13) \quad \partial I_1/\partial e = (n-2)e^{-1}I_1 - (n-2)e^{-2}I_0 + (n-2)e^{-2}I_{0-} - (n+1)(e-\dot{e})^{-1}I_1 + (n+1)(e-\dot{e})^{-2}I_0 - (n+1)(e-\dot{e})^{-2}I_{0+} ,$$

$$(14) \quad \partial I_1/\partial \dot{e} = (n+1)(e-\dot{e})^{-1}I_1 - (n+1)(e-\dot{e})^{-2}I_0 + (n+1)(e-\dot{e})^{-2}I_{0+} ,$$

etc.

The recurrence dependence for the derivatives of I_2, I_3 and I_4 is obvious.

$$(15) \quad \partial I_{0-}/\partial e = [e(1-e^2)]^{-1}\{ [3-2(n-1)e^2]I_{0-} + (n+1)(2e^2 - e\dot{e})I_{0+} - 3I_0 \} ,$$

$$(16) \quad \partial I_{0-}/\partial \dot{e} = (n+1)\dot{e}^{-1}(I_{0+} - I_{0-}) .$$

For $\partial I_{0+}/\partial e$ and $\partial I_{0+}/\partial \dot{e}$ we also have a linear dependence on I_0, I_{0-} and I_{0+}; for brevity we shall not write it here in an explicit form. For example, differentiating with respect to e and $\dot{e}$ the linear relation between $I_0(e, \dot{e}, n)$, $I_{0-}(e, \dot{e}, n)$ and $I_{0+}(e, \dot{e}, n)$, replacing the derivatives and also the integrals which differ from I_{0-} and I_{0+}, we shall obtain a linear homogeneous system for the later two integrals $I_{0-}(e, \dot{e}, n)$ and $I_{0+}(e, \dot{e}, n)$. For our purposes, it is enough to use not the full solution, but only the proportionality relation between I_{0-} and I_{0+}:

$$(17) \quad I_{0-}(e, \dot{e}, n) = D_{0-}(e, \dot{e}, n)I_{0+}(e, \dot{e}, n) ,$$

where $D_{0-}(e, \dot{e}, n)$ is a polynomial in e, $\dot{e}$ and n. It should be stressed that in the above linear relation there is not a free term, which follows from the homogeneity of the above mentioned system. Returning to the expressions I_0, I_1, I_2 and I_4, which also do not include free terms (in the sense, terms in which absent integrals of the type (5) - (7)), and replacing consecutively the results for I_{0-} , I_0 , I_1 , I_2 , we shall obtain proportionality relations of the same type as (17). There is, however, a gap in our solution for the system of integrals (5) - (7), because we have not found yet any expression for the integral I_3 . We can differentiate $I_3(e, \dot{e}, n)$ with respect to e or $\dot{e}$. In the later case the result is a differential equation for I_3 which takes a simple form :

$$(18) \qquad \partial[I_3 + (e - \dot{e})^{-1} I_2]/\partial\dot{e} = (n + 1)(e - \dot{e})^{-1} [I_3 + (e - \dot{e})^{-1} I_2] - n(e - \dot{e})^{-2} I_2$$

This equation enables us to find $I_3(e, \dot{e}, n)$ if the expression for $I_2(e, \dot{e}, n)$ is already known. It may be checked that a proportionality relation $I_3(e, \dot{e}, n) = D_3(e, \dot{e}, n)I_{0+}(e, \dot{e}, n)$, where D_3 contains a polynomial part in e, $\dot{e}$, n, and, additionally, exponential function of e, $\dot{e}$ and n, may serve as a general solution of the equation (18). Summarising all the results, we see that 6 of the integrals (5) - (7) are expressible through the seventh one:

$$(19) \qquad I_k(e, \dot{e}, n) = D_k(e, \dot{e}, n)I_{0+}(e, \dot{e}, n) , \qquad k = 0-, 0, 1, \ldots , 4 .$$

$D_k(e, \dot{e}, n)$ are already known functions, containing polynomials in e, $\dot{e}$, n and exponential functions depending also on e, $\dot{e}$ and n. The dynamical equation (8) then becomes into the form:

$$(20) \qquad \{ \sum[A_{ik} (e, \dot{e}, n)\ddot{e} + B_{ik} (e, \dot{e}, n)]D_i (e, \dot{e}, n)D_k(e, \dot{e}, n) \} (I_{0+} (e, \dot{e}, n))^2 = 0 ,$$

where the sum is over i and k (i, k = 0-, 0+, 0, 1, . . . , 4), and i is less or equal to k. Taking into account that the integral $I_{0+}(e, \dot{e}, n)|_{|e-\dot{e}|<1}$ is always strictly positive, it is possible to cancel out $(I_{0+})^2$ and to rewrite (20) as:

$$(21) \qquad [\sum\underline{A}_{ik} (e, \dot{e}, n)]\ddot{e} + \sum\underline{B}_{ik} (e, \dot{e}, n) = 0 , \qquad i, k = 0-, 0+, 0, 1, \ldots , 4 ,$$

where the both sums are again over i and k, and also i is less or equal to k. Here $\underline{A}_{ik}$ and $\underline{B}_{ik}$ are polynomials or exponential functions in e, $\dot{e}$ and n. Numerical integration of the equation (21) does not already require any computation of integrals (which include also the unknown solution $e(u)$ and its derivative $\dot{e}(u)$). So, the situation concerning solution of equation (1) is improved at least in order to simplify the computational procedure. Possibly, equation (21) admits applicability of more stable algorithms in order to find more accurate solution of the problem. Is it possible to obtain an analytical solution to the simplified equation (21) is still an open question, which is under investigation.

R e f e r e n c e s

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ЕДНО ВЪЗМОЖНО ОПРОСТЯВАНЕ НА ДИНАМИЧНОТО УРАВНЕНИЕ, ЗАДАВАЩО ЕВОЛЮЦИЯТА НА ЕЛИПТИЧНИТЕ АКРЕЦИОННИ ДИСКОВЕ

Димитър Димитров

Р е з ю м е

Показано е, че при допускането на закон за вискозитета от вида $\eta = \beta\Sigma^{n}$, интегралите, включени в уравнението което описва динамиката на диска, могат да бъдат заместени с полиноми и експоненциални функции от ексцентрицитета e, неговата производна $\dot{e} = \partial e/\partial(ln\ p)$ спрямо фокалния параметър p и от степенния показател n. Тази трансформация е полезна при численото решаване на динамичното диференциално уравнение от втори ред, защото тя допуска да се избегне численото оценяване на интегралите и евентуално обуславя по-стабилна изчислителна процедура. Нашето разглеждане на задачата е ограничено до случая, когато стойностите на параметъра n не са цели числа.